XX. *A new Method of inveſtigating the Sums of Infinite Series.* *By the Rev.* Samuel Vince, *A. M. F. R. S.*

Read June 2, 1791.

THE ſummation of infinite ſeries is a ſubject, not only of curious ſpeculation, but alſo of the greateſt importance in the various branches of mathematics and philoſophy; in conſequence of which it has always claimed a very conſiderable ſhare of attention from the moſt celebrated mathematicians. I ſhall therefore make no apology for offering to the public the following new and very expeditious method, by which we may obtain the ſums of a great variety of ſeries, moſt of which have never before been treated of. As the ſummation depends on the ſums of the reciprocals of the powers of the natural numbers, tables of ſuch ſums are given as far as the 40th power to twelve places of decimals, by which the ſums of the ſeries will be found true to ten or eleven places; and if greater accuracy were required (which is a caſe that can very rarely happen) it might eaſily be obtained by continuing the tables. The firſt and third columns ſhew the ſums, and the ſecond and fourth the powers correſponding.

TABLE I.

Sum of $\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+$ &c. *ad infinitum.*

Sum	$n=$	Sum	$n=$
A =,644934066848	2	W =,000000238450	22
B =,202056903159	3	X =,000000119219	23
C =,082323233711	4	Y =,000000059608	24
D =,036927755107	5	Z =,000000029803	25
E =,017343061984	6	A′ =,000000014901	26
F =,008349277387	7	B′ =,000000007450	27
G =,004077356198	8	C′ =,000000003725	28
H =,002008392826	9	D′ =,000000001863	29
I =,000994575128	10	E′ =,000000000931	30
K =,000494188604	11	F′ =,000000000465	31
L =,000246086553	12	G′ =,000000000233	32
M =,000122713347	13	H′ =,000000000116	33
N =,000061248135	14	I′ =,000000000058	34
O =,000030588236	15	K′ =,000000000029	35
P =,000015282259	16	L′ =,000000000015	36
Q =,000007637196	17	M′ =,000000000007	37
R =,000003817292	18	N′ =,000000000004	38
S =,000001908212	19	O′ =,000000000002	39
T =,000000953961	20	P′ =,000000000001	40
V =,000000476932	21		

TABLE

TABLE II.

Sum of $\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} +$ &c. *ad infinitum.*

Sum	$n =$	Sum	$n =$
a = ,177532966576	2	w = ,000000238386	22
b = ,098457322630	3	x = ,000000119199	23
c = ,052967170503	4	y = ,000000059602	24
d = ,027880229587	5	z = ,000000029801	25
e = ,014448908703	6	a' = ,000000014901	26
f = ,007406180072	7	b' = ,000000007450	27
g = ,003766998147	8	c' = ,000000003725	28
h = ,001905702459	9	d' = ,000000001863	29
i = ,000960492403	10	e' = ,000000000931	30
k = ,000482856502	11	f' = ,000000000465	31
l = ,000242314856	12	g' = ,000000000233	32
m = ,000121457237	13	h' = ,000000000116	33
n = ,000060829654	14	i' = ,000000000058	34
o = ,000030448787	15	k' = ,000000000029	35
p = ,000015235790	16	l' = ,000000000015	36
q = ,000007621708	17	m' = ,000000000007	37
r = ,000003812130	18	n' = ,000000000004	38
s = ,000001906491	19	o' = ,000000000002	39
t = ,000000953389	20	p' = ,000000000001	40
v = ,000000476742	21		

TABLE III.

Sum of $\frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} +$ &c. *ad infinitum.*

Sum	n =	Sum	n =
A″ = ,411233516712	2	W″ = ,000000238419	22
B″ = ,150257112895	3	X″ = ,000000119209	23
C″ = ,067645202107	4	Y″ = ,000000059605	24
D″ = ,032403992347	5	Z″ = ,000000029802	25
E″ = ,015895985344	6	A‴ = ,000000014901	26
F″ = ,007877728730	7	B‴ = ,000000007450	27
G″ = ,003922177173	8	C‴ = ,000000003725	28
H″ = ,001957047643	9	D‴ = ,000000001863	29
I″ = ,000977533765	10	E‴ = ,000000000931	30
K″ = ,000488522553	11	F‴ = ,000000000465	31
L″ = ,000244200705	12	G‴ = ,000000000233	32
M″ = ,000122085292	13	H‴ = ,000000000116	33
N″ = ,000061038895	14	I‴ = ,000000000058	34
O″ = ,000030518512	15	K‴ = ,000000000029	35
P″ = ,000015259024	16	L‴ = ,000000000015	36
Q″ = ,000007629452	17	M‴ = ,000000000007	37
R″ = ,000003814712	18	N‴ = ,000000000004	38
S″ = ,000001907352	19	O‴ = ,000000000002	39
T″ = ,000000953675	20	P‴ = ,000000000001	40
V″ = ,000000476837	21		

TABLE IV.

Sum of $\frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} +$ &c. *ad infinitum.*

Sum	$n =$	Sum	$n =$
$a'' = ,233700550136$	2	$n'' = ,000000209240$	14
$b'' = ,051799790264$	3	$o'' = ,000000069724$	15
$c'' = ,014678031604$	4	$p'' = ,000000023234$	16
$d'' = ,004523762760$	5	$q'' = ,000000007744$	17
$e'' = ,001447076640$	6	$r'' = ,000000002581$	18
$f'' = ,000471548657$	7	$s'' = ,000000000864$	19
$g'' = ,000155179025$	8	$t'' = ,000000000286$	20
$h'' = ,000051345183$	9	$v'' = ,000000000095$	21
$i'' = ,000017041362$	10	$w'' = ,000000000032$	22
$k'' = ,000005666051$	11	$x'' = ,000000000011$	23
$l'' = ,000001885848$	12	$y'' = ,000000000004$	24
$m'' = ,000000628055$	13	$z'' = ,000000000001$	25

PROP. I.

To find the ſum of the ſums of the reciprocal ſquares, cubes, &c. &c. ad infinitum.

By diviſion $\frac{1}{x-1 \times x} = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} +$ &c. *ad inf.*; hence if we make each of theſe terms the general term of a ſeries, and write 2, 3, 4, &c. *ad inf.* for x, we have $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} +$ &c. = (Table 1.) $A + B + C + D +$ &c.; but $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} +$ &c. *ad inf.* $= 1$; hence $A + B + C + D +$ &c. *ad inf.* $= 1$.

As $\frac{1}{x \times \overline{x+1}} = \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} - \frac{1}{x^5} + \&c.$ *ad inf.*; we have, by the ſame method of proceeding, $A - B + C - D + \&c.$ *ad inf.* $= \frac{1}{2}$; conſequently $A + C + E + \&c. = \frac{3}{4}$, and $B + D + F + \&c. = \frac{1}{4}$.

Becauſe $\frac{1}{\overline{x-1} \times x} = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \&c.$ *ad inf.*; if for x we write 2, 4, 6, &c., then will $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c. =$ (Tab. 3) $A'' + B'' + C'' + D'' + \&c.$; but $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c. =$ hyp. log. 2; hence $A'' + B'' + C'' + D'' + \&c. =$ hyp. log. 2.

If in the ſame expreſſion we write 3, 5, 7, &c. for x, then $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \&c. =$ (Tab. 4.) $a'' + b'' + c'' + \&c.$; but $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \&c. = 1 -$ hyp. log. 2; hence $a'' + b'' + c'' + \&c. = 1 -$ hyp. log. 2.—Hence from either of theſe two laſt caſes, we have a very expeditious method of finding the hyp. log. 2.

PROP. II.

To find the ſum of the infinite ſeries whoſe general term is $\frac{1}{mx^r \pm n}$.

By diviſion $\frac{1}{mx^r \pm n} = \frac{1}{mx^r} \mp \frac{n}{m^2x^{2r}} + \frac{n^2}{m^3x^{3r}} \mp \frac{n^3}{m^4x^{4r}} + \&c.$ *ad inf.*; hence, if $\frac{1}{mx^r \pm n}$ be made the general term of a ſeries, and for x we write 2, 3, 4, &c., its ſum will be equal to the ſums of another ſet of ſerieſes, whoſe terms are the powers of the reciprocals of the natural numbers reſpectively multiplied into

into $\frac{1}{m}$, $\frac{n}{m^2}$, $\frac{n^2}{m^3}$, &c.; hence the ſum of each of theſe ſeries being known from the tables, the ſum of the given ſeries will be found.

Ex. 1. Let $\frac{1}{x^2+1}$ be the general term; now $\frac{1}{x^2+1}=\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}+\&c.$; hence if for x we write 2, 3, 4, &c. we have $\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\&c.=A-C+E-G+\&c.=$ (by Tab. 1.) ,576674037469.

Ex. 2. Let $\frac{1}{x^2-1}$ be the general term; then, by the ſame method of proceeding, $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\&c.=A+C+E+\&c.=$ (by Prop. 1.) $\frac{3}{4}$.

Cor. Becauſe $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\&c.=\frac{1}{8}\times\overline{1+\frac{1}{3}+\frac{1}{6}+\&c.}=$ (as $1+\frac{1}{3}+\frac{1}{6}+\&c.$ is the reciprocal of the figurative numbers of the ſecond order) $\frac{1}{8}\times 2=\frac{1}{4}$; therefore $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\&c.=\frac{1}{2}$. Alſo, as $\frac{1}{x^2-1}=\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\&c.$; if we write 2, 4, 6, &c. for x, we have $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\&c.=$ (by Tab. 3.) $A''+C''+E''+\&c.=\frac{1}{2}$; but, by Prop. 1. $A''+B''+C''+D''+\&c.=$ hyp. log. 2; hence $B''+D''+F''+\&c.=-\frac{1}{2}+$ hyp. log. 2.

Ex. 3. Let the general term be $\frac{1}{x^3-1}=\frac{1}{x^3}+\frac{1}{x^6}+\frac{1}{x^9}+\&c.$, and, by writing 2, 3, 4, &c. for x, we have $\frac{1}{7}+\frac{1}{26}+\frac{1}{63}+\&c.=B+E+H+\&c.=$,221689395104.

Ex. 4. Let the general term be $\frac{1}{3x^4-2}=\frac{1}{3x^4}+\frac{2}{9x^8}+\frac{4}{27x^{12}}+$ &c., and, by writing 2, 3, 4, &c. for x, &c. we have $\frac{1}{46}+\frac{1}{241}+\frac{1}{766}+$ &c. $\frac{1}{3}C+\frac{2}{9}G+\frac{4}{27}L+$ &c. $=,028385252052$.

Ex. 5. To find the ſum of the ſeries $\frac{1}{9}-\frac{1}{26}+\frac{1}{65}-\frac{1}{124}+$ &c. If we write 2, -3, 4, -5, &c. for x, the general term will be $\frac{1}{x^3+1}=\frac{1}{x^3}-\frac{1}{x^6}+\frac{1}{x^9}-\frac{1}{x^{12}}+$ &c. Now, by writing 2, -3, 4, -5, &c. for x, the ſerieſes of which $\frac{1}{x^3}$, $\frac{1}{x^9}$, &c. are the general terms, will be alternately + and −, and therefore their ſums will be found in Tab. 2. and the ſerieſes of which $\frac{1}{x^6}$, $\frac{1}{x^{12}}$, &c. are the general terms will have their terms all +, and therefore their ſums will be found in Tab. 1. Hence the ſum required $= b+h+o+$ &c. $-E-L-R-$ &c. $=,082800931803$.

PROP. III.

To find the ſum of the ſums of the reciprocals of the odd *powers in* Tab. 2.

By diviſion $\frac{1}{\overline{x-1}\times x}=\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+$ &c.; hence by writing 2, -3, 4, -5, &c. for x, the ſums of the ſerieſes of which $\frac{1}{x^3}$, $\frac{1}{x^5}$, &c. are the general terms, may be found by Tab. 2. and the other ſums by Tab. 1.; hence $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+$ &c. $=A+C+E+$ &c. $+b+d+f+$ &c.; but $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+$ &c.

&c $= -\frac{1}{2} + 2$ hyp. log. 2; and by Prop. 1. $A + C + E +$ &c. $=$ $\frac{3}{4}$; hence $b + d + f +$ &c. $= -\frac{5}{4} + 2$ hyp. log. 2.

PROP. IV.

To find the ſum of the infinite ſeries whoſe general term is $\frac{x^s}{mx^r \pm n}$.

By diviſion $\frac{x^s}{mx^r \pm n} = \frac{1}{mx^{r-s}} \mp \frac{n}{m^2 x^{2r-s}} + \frac{n^2}{m^3 x^{3r-s}} \mp$ &c. *ad inf.*; hence the ſum of the ſeries of which $\frac{x^s}{mx^r \pm n}$ is the general term, is found as in Prop. 2. Here r muſt be greater than s at leaſt by 2, otherwiſe the ſum will be infinite.

Ex. 1. Let the general term be $\frac{x^2}{x^4+1} = \frac{1}{x^2} - \frac{1}{x^6} + \frac{1}{x^{10}} -$ &c.; hence if for x we write 2, 3, 4, &c. we have $\frac{4}{17} + \frac{9}{82} + \frac{16}{257} +$ &c. $= A - E + I - N +$ &c. $= ,538527924723$.—If for x we write 2, 4, 6, &c. we get $\frac{4}{17} + \frac{16}{257} + \frac{36}{1296} +$ &c. $= A'' - E'' + I'' - N'' +$ &c. $= ,396257616555$.

Ex. 2. Let the general term be $\frac{x}{3x^3-1} = \frac{1}{3x^2} + \frac{1}{9x^5} + \frac{1}{27x^8} +$ &c.; hence if we write 2, 3, 4, &c. for x, we have $\frac{2}{23} + \frac{3}{80} + \frac{4}{191} +$ &c. $=$ $\frac{1}{3}A + \frac{1}{9}D + \frac{1}{27}G +$ &c. $= ,219238483448$.

By this propoſition we may find the ſum of any ſeries whoſe general term is $\frac{ax^s + bx^{s-1} + cx^{s-2} + \text{\&c.}}{mx^r \pm n}$; for this reſolves itſelf into $\frac{ax^s}{mx^r \pm n}$, $\frac{bx^{s-1}}{mx^r \pm n}$, &c. &c., the ſum of each of which ſeries is found by this propoſition. Now the $\overline{s+1}$th differences of the numerators of this general term are $=0$, and therefore it comprehends all ſeries under ſuch circumſtances. For example, let the given ſeries be $\frac{4}{17}+\frac{13}{82}+\frac{26}{257}+\frac{43}{626}$. Here the third differences of the numerators $=0$; to find therefore the general expreſſion for the numerator, aſſume ax^2+bx+c for it; and, by writing 2, 3, 4, for x, we have $4a+2b+c=4$, $9a+3b+$ $-c=13$, $16a+4b+c=26$; hence $a=2$, $b=-1$, $c=-2$; and as the denominator is manifeſtly x^4+1, the general term will be $\frac{2x^2-x-2}{x^4+1}=\frac{2x^2}{x^4+1}-\frac{x}{x^4+1}-\frac{2}{x^4+1}$, each of which being made the general term of a ſeries, their ſum will be found to be reſpectively 1,077055849446, 0,194173022145, and 0,156955159332; hence the ſum of the given ſeries is 0,725927667969.

If s be negative, the general term becomes $\frac{1}{x^s\times\overline{mx^r\pm n}}=\frac{1}{mx^{r+s}}$ $\mp\frac{1}{m^2x^{2r+s}}+\frac{1}{m^3x^{3r+s}}\mp$ &c.

Ex. 1. To find the ſum of $\frac{1}{1\cdot 2\cdot 3}-\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}-$ &c. *ad inf.* Here the general term is $\frac{1}{\overline{x-1}\times x\times\overline{x+1}}=\frac{1}{x\times\overline{x^2-1}}=$ $\frac{1}{x^3}+\frac{1}{x^5}+\frac{1}{x^7}+$&c.; hence, by writing 2, -3, 4, -5, &c. for x, we have the ſum $=b+a+f+$ &c. $=$ (by Prop. 3.) $-\frac{5}{4}+2$ hyp. log. 2.

If $\frac{1}{\overline{x-1}\times x^3\times\overline{x+1}}$ be the general term it reſolves itſelf into $\frac{1}{x^5}+\frac{1}{x^7}+\frac{1}{x^9}+$&c.; conſequently the ſum of $\frac{1}{1\cdot 2^3\cdot 3}-\frac{1}{2\cdot 3^3\cdot 4}$

+

$+\frac{1}{3\cdot4^{3}\cdot5}-$ &c. $=-b-\frac{5}{4}+2$ hyp. log. 2. In like manner the ſum of $\frac{1}{1\cdot2^{5}\cdot3}-\frac{1}{2\cdot3^{5}\cdot4}+\frac{1}{3\cdot4^{5}\cdot5}-$ &c. $=-b-d-\frac{5}{4}+$ 2 hyp. log. 2. Thus we may proceed as far as we pleaſe by adding two powers to the middle term; and hence this remarkable property of the ſerieſes, that the difference of the ſums of the ſerieſes where the middle term is x, x^{3}, x^{5}, &c. is b, d, f, &c. reſpectively.

Ex. 2. In like manner if the general term be $\frac{1}{\overline{x-1}\times x^{3}\times\overline{x+1}}$, and we write 2, 3, 4, &c. for x, we have $\frac{1}{1\cdot2^{3}\cdot3}+\frac{1}{2\cdot3^{3}\cdot3}+\frac{1}{3\cdot4^{3}\cdot5}+$ &c. $=\text{D}+\text{F}+\text{H}+$ &c. $=$ (by Prop. 1.) $\frac{1}{4}-\text{B}$. Hence alſo $\frac{1}{1\cdot2^{5}\cdot3}+\frac{1}{2\cdot3^{5}\cdot4}+$ &c. $=\frac{1}{4}-\text{B}-\text{D}$; and ſo on as before.

If the general term be under the form $\frac{1}{x^{n}\cdot\overline{x+m}}$, it will be moſt convenient to reſolve it thus: by diviſion $\frac{1}{x+m}=\frac{1}{x}-\frac{m}{x^{2}}+\frac{m^{2}}{x^{3}}-$ &c. $\pm\frac{m^{n}}{x^{n}\cdot\overline{x+m}}$; hence $\pm\frac{1}{x^{n}\cdot\overline{x+m}}=$ $\overline{\frac{1}{x+m}-\frac{1}{x}+\frac{m}{x^{2}}-\frac{m^{2}}{x^{3}}+\text{\&c.}}\times\frac{1}{m^{n}}=\overline{-\frac{m}{x\cdot\overline{x+m}}+\frac{m}{x^{2}}-\frac{m^{2}}{x^{3}}+\text{\&c.}}\times\frac{1}{m^{n}}$, where the ſign on the left hand will be + or − according as n is even or odd, and the number of terms on the right is $=n$. Now the ſum of the ſeries whoſe general term is $\frac{m}{x\cdot\overline{x+m}}$ is well known, and the ſums of the other are found from the tables.

Ex. 1. To find the ſum of $\frac{1}{2^{2}\cdot3}+\frac{1}{3^{2}\cdot4}+\frac{1}{4^{2}\cdot5}+$ &c. *ad inf.*

Here

Here the general term is $\frac{1}{x^2\times\overline{x+1}}=-\frac{1}{x\,.\,\overline{x+1}}+\frac{1}{x^2}$, and by writing 2, 3. 4, &c. for x, we have the ſum $=-\frac{1}{2\cdot3}-\frac{1}{3\cdot4}-$ &c. $+\mathrm{A}=-\frac{1}{2}+\mathrm{A}$. In like manner $\frac{1}{2^3\cdot3}+\frac{1}{4^2\cdot5}+\frac{1}{6\cdot7}+$ &c. $=-1+$ hyp.log. $2+\mathrm{A}''$. Alſo $\frac{1}{2^3\cdot5}+\frac{1}{3^3\cdot6}+\frac{1}{4^3\cdot7}+$ &c. $=\overline{\frac{13}{12}-3\mathrm{A}+9\mathrm{B}}\times\frac{1}{27}$.

If m be negative, then $\frac{1}{x^n\cdot\overline{x-m}}=\overline{\frac{m}{x\,.\,\overline{x-m}}-\frac{m}{x^2}-\frac{m^2}{x^3}-\text{\&c.}}\times\frac{1}{m^n}$. Hence $\frac{1}{2^4\,.\,1}+\frac{1}{3^4\,.\,2}+\frac{1}{4^4\cdot3}+$ &c. $=1-\mathrm{A}-\mathrm{B}-\mathrm{C}$; and ſo on for others of the ſame kind.

If the general term be under this form $\frac{1}{x^{rn}\,.\,\overline{ax^n+m}}$, then, in like manner, we have $\pm\frac{1}{x^{rn}\,.\,\overline{ax^n+m}}=\overline{\frac{1}{ax^n+m}-\frac{1}{ax^n}+\frac{m}{a^2x^{2n}}-\text{\&c.}}\times\frac{a^r}{m^r}$, where the ſign on the left hand will be + or −, according as r is even or odd, and the number of terms on the right is $=r+1$.

Ex. 1. To find the ſum of $\frac{1}{2^4\cdot5}+\frac{1}{3^4\,.\,10}+\frac{1}{4^4\cdot17}+$ &c. Here $m=1$, $n=2$, $r=2$, $a=1$, and the general term $\frac{1}{x^4\times\overline{x^2+1}}=\frac{1}{x^2+1}-\frac{1}{x^2}+\frac{1}{x^4}$; now the ſum of the ſeries whoſe general term is $\frac{1}{x^2+1}$ is $=,576674037469$, by Prop. 2.; conſequently the ſum required $=,576674037469-\mathrm{A}+\mathrm{C}=,014063204332$.

Ex. 2. If the given ſeries be $\frac{1}{4\cdot5}+\frac{1}{9\,.\,10}+\frac{1}{16\,.\,17}+$ &c. the general

general term will be $\frac{1}{x^2\cdot\overline{x+1}} = -\frac{1}{x^2+1}+\frac{1}{x^2}$; hence, by writing 2, 3, 4, &c. for x, we have the ſum $= -,576674037469 + A = ,06826002938$.

If m be negative, then $\frac{1}{x^{rn}\cdot\overline{ax^n-m}} = \overline{\frac{1}{ax^n-m} - \frac{1}{ax^n} - \frac{m}{a^2x^{2n}} - \&c.} \times \frac{a^r}{m^r}$.

Ex. 1. To find the ſum of $\frac{1}{1\cdot 2^2\cdot 3}+\frac{1}{2\cdot 3^2\cdot 4}+\frac{1}{3\cdot 4^2\cdot 5}+$ &c. Here the general term is $\frac{1}{\overline{x-1}\times x^2\times\overline{x+1}} = \frac{1}{x^2\times\overline{x^2-1}} = \frac{1}{x^2-1}-\frac{1}{x^2}$; now, by writing 2, 3, 4, &c. for x, the ſum of the ſeries whoſe general term is $\frac{1}{x^2-1}$ is $=\frac{3}{4}$, by Prop. 2.; hence the ſum required $=\frac{3}{4}-A$.

Ex. 2. Let the given ſeries be $\frac{1}{1\cdot 2^2\cdot 3}+\frac{1}{3\cdot 4^2\cdot 5}+\frac{1}{5\cdot 6^2\cdot 7}+$ &c. Here the general term is the ſame as before, writing 2, 4, 6, &c. for x; and, by Prop. 2. the ſum of the ſeries whoſe general term is $\frac{1}{x^2-1}$ is $=\frac{1}{2}$; hence the ſum $=\frac{1}{2}-A''$.

Ex. 3. In like manner the ſum of the ſeries $\frac{1}{1\cdot 2^4\cdot 3}+\frac{1}{2\cdot 3^4\cdot 4}+\frac{1}{3\cdot 4^4\cdot 5}+$ &c. $= ,221689395104 - B$.

Ex. 4. To find the ſum of $\frac{1}{3\cdot 4^2\cdot 5}+\frac{1}{8\cdot 9^2\cdot 10}+\frac{1}{15\cdot 16^2\cdot 17}$ + &c. Here the general term is $\frac{1}{\overline{x^2-1}\times x^4\times\overline{x^2+1}} = \frac{1}{x^4\times\overline{x^4-1}} = \frac{1}{x^4-1}-\frac{1}{x^4}$; but the ſum of the ſeries whoſe general term is $\frac{1}{x^4-1}$ is $= ,086662976264$; hence the ſum required $= ,086662976264 - C$.

PROP.

PROP. V.

To find the ſum of the infinite ſeries $\frac{1}{15}+\frac{1}{40}+\frac{1}{85}+\frac{1}{156}+\frac{1}{259}+$ &c.

In this ſeries the fourth differences of the denominators $=0$; therefore the general term muſt be repreſented by $\frac{1}{ax^3+bx^2+cx+d}$; write therefore 2, 3, 4, &c. for x, and we have $8a+4b+2c+d=15$, $27a+9b+3c+d=40$, $64a+16b+4c+d=85$, $125a+25b+5c+d=156$; hence $a=1$, $b=1$, $c=1$, $d=1$, and the general term is $\frac{1}{x^3+x^2+x+1}=\frac{1}{x^3}-\frac{1}{x^4}+\frac{1}{x^7}-\frac{1}{x^8}+$ &c.; hence the ſum $=\text{B}-\text{C}+\text{F}-\text{G}+\text{K}-\text{L}+$ &c. $=$,1242700165.

PROP. VI.

To find the ſum of $\frac{2}{5^2}+\frac{3}{10^2}+\frac{4}{17^2}+$ &c. *ad inf.*

The general term $=\frac{x}{\overline{x^2+1}^2}=\frac{1}{x^3}-\frac{2}{x^5}-\frac{3}{x^7}+$ &c.; hence, by writing 2, 3, 4, &c. for x, we have the ſum $=\text{B}-2\text{D}+3\text{F}-$ &c. $=$,147115771469.

In like manner $\frac{2}{3^2}+\frac{3}{8^2}+\frac{4}{15^2}+$ &c. $=\text{B}+2\text{D}+3\text{F}+$ &c. $=$,312498999865.

PROP. VII.

To find the ſum of $\frac{1}{3^2 . 5^2}+\frac{1}{8^2 . 10^2}+\frac{1}{15^2 . 17^2}+$ &c. *ad inf.*

The general term is $\frac{1}{\overline{x^2-1}^2\times\overline{x^2+1}^2}=\frac{1}{x^8}+\frac{2}{x^{12}}+\frac{3}{x^{16}}+$ &c.; hence, by writing 2, 3, 4, &c. for x, we have the ſum $=$ $G+2L+3P+$ &c. $=$,009447690684.

PROP. VIII.

To find the ſum of $\frac{1}{1^3 . 2^3 . 3^3}+\frac{1}{2^3 . 3^3 . 4^3}+\frac{1}{3^3 . 4^3 . 5^3}+$ &c. *ad inf.*

Here the general term is $\frac{1}{\overline{x-1}^3 . x^3 . \overline{x+1}^3}=\frac{1}{x^9}+\frac{3}{x^{11}}+\frac{6}{x^{13}}+$ &c. and hence the ſum $=H+3K+6M+$ &c. $=$,004707148337.

PROP. IX.

To find the ſum of the infinite ſeries $1-\frac{1}{3}+\frac{1}{6}-\frac{1}{10}+$ &c. *being a ſeries of the reciprocal of the figurative numbers of the* 3rd *order, having the ſigns alternately* $+$ *and* $-$.

This ſeries, by reſolving two terms into one, becomes $\frac{4}{1 . 2 . 3}+\frac{4}{3 . 4 . 5}+\frac{4}{5 . 6 . 7}+$ &c. whoſe general term, by writing 2, 4, 6, &c. for x, is $\frac{4}{\overline{x-1}\times x\times\overline{x+1}}=\frac{4}{x^3}+\frac{4}{x^5}+\frac{4}{x^7}+$ &c. conſequently the ſum $=4B''+4D''+4F''+$ &c. $=$ (by Cor. Ex. 2. Prop. 2.) $-2+4$ hyp. log. 2.

Cor. Hence, as $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$&c. $=2$, we have $1+\frac{1}{6}+\frac{1}{15}+$&c. $=2$ hyp. log. 2, and $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+$ &c. $=2-2$ hyp. log. 2.

PROP. X.

To find the ſum of the infinite ſeries $1-\frac{1}{4}+\frac{1}{10}-\frac{1}{20}+$ &c. *being the reciprocals of the figurative numbers of the 4th order, having the ſigns alternately* + *and* −.

If we write 2, −3, 4, −5, &c. for x, the general term will be $\frac{6}{x^3-x}=\frac{6}{x^3}+\frac{6}{x^5}+\frac{6}{x^7}+$&c.; hence the ſum required $=6b+6d+6f+$&c. $=$ (by Prop. 3.) $-7\frac{1}{2}+12$ hyp. log. 2.

Cor. Becauſe the ſum of $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+$&c. $=\frac{3}{2}$; therefore $1+\frac{1}{10}+\frac{1}{35}+$&c. $=-3+6$ hyp. log. 2; and $\frac{1}{4}+\frac{1}{10}+\frac{1}{56}+$ &c. $=4\frac{1}{2}-6$ hyp. log. 2.

PROP. XI.

To find the ſum of $\frac{2^2}{1^2\cdot 3^2}+\frac{3^2}{2^2\cdot 4^2}+\frac{4^2}{3^2\cdot 5^2}+$&c. *ad infinitum.*

The general term, by writing 2, 3, 4, &c. for x, is $\frac{x^2}{\overline{x-1}^2\times\overline{x+1}^2}=\frac{1}{x^2}+\frac{2}{x^4}+\frac{3}{x^6}+$&c.; hence the ſum $=A+2C+3E$ $+$&c. $=,884966993407$.

PROP. XII.

To find the ſum of $\frac{1}{1\cdot 2^2\cdot 3^3}+\frac{1}{2\cdot 3^2\cdot 4^3}+\frac{1}{3\cdot 4^2\cdot 5^3}+$ &c. *ad infinitum.*

Here the general term, by writing 2, 3, 4, &c. for x, is $\frac{1}{\overline{x-1}\cdot x^2\cdot\overline{x+1}^3}=\frac{1}{x^6}-\frac{2}{x^7}+\frac{4}{x^8}+\frac{6}{x^9}-\frac{9}{x^{10}}-\frac{12}{x^{11}}+\frac{16}{x^{12}}-$ &c.; conſequently the ſum $= E-2F+4G-6H+9I-12K+$ &c. $=$,010370898482.

PROP. XIII.

To find the ſum of $\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C-$ &c. *ad infinitum.*

The hyp. log. $2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c. $=1+\frac{1}{3}+\frac{1}{5}+$ &c. $-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-$ &c.; hence $2\times$ hyp. log. 2, or hyp. log. 4, $=\frac{2}{1}+\frac{2}{3}+\frac{2}{5}+$ &c. $-1-\frac{1}{2}-\frac{1}{3}-$ &c. Now, by diviſion, $\frac{2}{2x+1}=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{4x^3}-\frac{1}{8x^4}+$ &c.; hence, by writing 2, 3, 4, &c. for x, we have (after tranſpoſition) $\frac{2}{5}+\frac{2}{7}+$ &c. $-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}$ $-$ &c. $=-\frac{1}{2}A+\frac{1}{4}B-\frac{1}{8}C+$ &c.; hence, by adding equal quantities to each ſide, we have $\frac{2}{1}+\frac{2}{3}+\frac{2}{5}+$ &c. $-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$ &c. $=\frac{8}{3}-\frac{1}{2}A+\frac{1}{4}B-\frac{1}{8}C+$ &c., conſequently $\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C-$ &c. $=\frac{8}{3}-\frac{2}{1}-\frac{2}{3}-\frac{2}{5}-$ &c. $+1+\frac{1}{2}+\frac{1}{3}+$ &c. $=\frac{8}{3}-$ hyp. log. 4.

PROP. XIV.

To find the ſum of the infinite ſeries $\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\frac{1}{4\cdot 9}+\&c.$

The general term, by writing 2, 3, 4, &c. for x, is $\frac{1}{x\,.\,\overline{2x+1}}=\frac{1}{2x^2}-\frac{1}{4x^3}+\frac{1}{8x^4}-\&c.$; hence the ſum $=\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C-\&c.=$ (by Prop. 13.) $\frac{8}{3}-$ hyp. log. 4.

PROP. XV.

To find the ſum of $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$ *to* $\frac{1}{x}$.

The hyp. log. $\frac{x}{x-1}=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\frac{1}{4x^4}+\&c.$; conſequently hyp. log. $\frac{x}{x-1}-\frac{1}{2x^2}-\frac{1}{3x^3}-\frac{1}{4x^4}-\&c.=\frac{1}{x}$; hence, if we write 2, 3, 4, &c. for x, we have hyp. log. $\frac{2}{1}+$ hyp. log. $\frac{3}{2}+\&c.\ldots\ldots$

$$\left.\begin{array}{l}\text{hyp. log. }\frac{x}{x-1}-\frac{1}{2}\times\overline{\frac{1}{2^2}+\frac{1}{3^2}+\&c.\ldots\ldots\frac{1}{x^2}}\\ \qquad\qquad\quad -\frac{1}{3}\times\overline{\frac{1}{2^3}+\frac{1}{3^3}+\&c.\ldots\ldots\frac{1}{x^3}}\\ \qquad\qquad\quad -\frac{1}{4}\times\overline{\frac{1}{2^4}+\frac{1}{3^4}+\&c.\ldots\ldots\frac{1}{x^4}}\\ \qquad\qquad\quad -\quad\&c.\quad\&c.\qquad\quad\&c.\end{array}\right\}=$$

$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\ldots\frac{1}{x}$; but hyp. log. $\frac{2}{1}+$ hyp. log. $\frac{3}{2}+$ hyp. log. $\frac{4}{3}+\&c.\ldots\ldots$ hyp. log. $\frac{x}{x-1}=$ hyp. log. $\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\&c.\ldots\ldots\frac{x}{x-1}=$ hyp. log. x; alſo $\frac{1}{2^2}+\frac{1}{3^2}+\&c.\ldots\ldots\frac{1}{x^2}=$ the ſum of the ſame ſeries *ad infinitum*, *minus* the ſum of all the terms from $\frac{1}{x^2}$ = (if $x+1=n$) $A-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{6n^3}+\frac{1}{30n^5}-\frac{1}{42n^7}+\&c.$; in the ſame manner $\frac{1}{2^3}+\frac{1}{3^3}+\&c.\dots\frac{1}{x^3}=B-\frac{1}{2n^2}-\frac{1}{2n^3}-\frac{1}{4n^4}+\frac{1}{12n^6}-\frac{1}{12n^8}+\&c.$; and ſo on for the other ſerieſes; hence, by ſubſtitution, and adding unity to each ſide, we have hyp. log. $x+1-\frac{1}{2}A-\frac{1}{3}B-\frac{1}{4}C-\&c.+\frac{1}{2n}+\frac{5}{12n^2}+\frac{1}{3n^3}+\frac{31}{120n^4}+\frac{1}{5n^5}+\frac{41}{252n^6}+\frac{1}{7n^7}+\frac{31}{240n^8}+\&c.=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\dots\frac{1}{x}$; but $1-\frac{1}{2}A-\frac{1}{3}B-\frac{1}{4}C-\&c.=,577215664901$; hence $1+\frac{1}{2}+\frac{1}{3}+\&c.\dots\frac{1}{x}=$ hyp. log. $x+,577215664901+\frac{1}{2n}+\frac{5}{12n^2}+\frac{1}{3n^3}+\frac{31}{120n^4}+\frac{1}{5n^5}+\frac{41}{252n^6}+\frac{1}{7n^7}+\frac{31}{240n^8}+\&c.$

Ex. 1. Let $x=10000$; then

hyp. log. 10000 = 9,210340371976
conſt. quant. = ,577215664901
$\frac{1}{2n}$ = ,000049995000
$\frac{5}{12n^2}$ = ,000000004166

therefore the ſum required = 9,787606036043

Ex. 2. Let $x=10000000$; then

hyp. log. 10000000 = 16,118095650958
conſt. quant. = ,577215664901
$\frac{1}{2n}$ = ,000000049999

therefore the ſum required = 16,695311365858

PROP. XVI.

To find the value of $\alpha \times \beta \times \gamma \times \delta \times$ &c. *ad infinitum, ſuppoſing the general term to be a rational function of* x.

Let π be the general term, then reſolve $\frac{\dot{\pi}}{\pi}$ into an infinite ſeries, and take the fluent on both ſides; then write 2, 3, 4, &c. for x, and one ſide will become the hyp. log. of the given ſeries, and the value of the other ſide may be found from the tables.

Ex. 1. To find the value of $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^2}{x^2-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{2\dot{x}}{x^3-x} = -\frac{2\dot{x}}{x^3} - \frac{2\dot{x}}{x^5} - \frac{2\dot{x}}{x^7} -$ &c.; hence the hyp. log. $\frac{x^2}{x^2-1} = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} +$ &c. Write 2, 3, 4, &c. for x, and we have the hyp. log. $\frac{4}{3}$ + hyp. log. $\frac{9}{8}$ + hyp. log. $\frac{16}{15}$ + &c. $= A + \frac{1}{2}C + \frac{1}{3}E +$ &c. = ,693147180574, which is the hyp. log. 2; but hyp. log. $\frac{4}{3}$ + hyp. log. $\frac{9}{8}$ + hyp. log. $\frac{16}{15}$ + &c. = hyp. log. $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. conſequently $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. = 2.

Ex. 2. To find the value of $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^3}{x^3-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{3\dot{x}}{x^4-x} = -\frac{3\dot{x}}{x^4} - \frac{3\dot{x}}{x^7} - \frac{3\dot{x}}{x^{10}} -$ &c.; hence the hyp. log. $\frac{x^3}{x^3-1} = \frac{1}{x^3} + \frac{1}{2x^6} + \frac{1}{3x^9} +$ &c. Write 2, 3, 4, &c. for x, and we have hyp. log. $\frac{8}{7}$ + hyp. log.

$\frac{27}{26}$

$\frac{27}{26}$ + hyp. log. $\frac{64}{63}$ + &c. = B + $\frac{1}{2}$E + $\frac{1}{3}$H + &c. = ,211466250444; or hyp. log. $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. = ,211466250444; hence $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. = 1,627295, &c.

Hence we may find the value of ſuch a quantity, ſuppoſing the number of factors to be finite.

Ex. To find the value of $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ &c. . . . . to $\frac{2x}{2x-1}$.

Here the general term being $\frac{2x}{2x-1}$, we have $\frac{\dot{\pi}}{\pi} = -\frac{\dot{x}}{2x^2 - x} = -\frac{\dot{x}}{2x^2} - \frac{\dot{x}}{4x^3} - \frac{\dot{x}}{8x^4} - \frac{\dot{x}}{16x^5}$ − &c.; hence hyp. log. $\frac{2x}{2x-1} = \frac{1}{1 \cdot 2x} + \frac{1}{2 \cdot 4x^2} + \frac{1}{3 \cdot 8x^3} + \frac{1}{4 \cdot 16x^4}$ + &c.. Now write 2, 3, 4, &c. for x, and we have the hyp. log. $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. . . . .

$$\frac{2x}{2x-1} = \left.\begin{array}{l} \frac{1}{2} \times \overline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{x}} \\ + \frac{1}{2 \cdot 4} \times \overline{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots \frac{1}{x^2}} \\ + \frac{1}{3 \cdot 8} \times \overline{\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\ldots \frac{1}{x^3}} \\ + \quad \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \end{array}\right\}$$

But, by Prop. 15. $\frac{1}{2} \times \overline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{x}} = \frac{1}{2}$hyp. log. $x - ,211392167549 + \frac{1}{4n} + \frac{5}{24n^2} + \frac{1}{6n^3} + \frac{31}{240n^4} + \frac{1}{10n^5} + \frac{41}{504n^6}$ + &c.; alſo $\frac{1}{2 \cdot 4} \times \overline{\frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots \frac{1}{x^2}} = \frac{1}{2 \cdot 4} \times \overline{A - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5} - \text{\&c.}}$ and $\frac{1}{3 \cdot 8} \times \overline{\frac{1}{2^3} + \frac{1}{3^3} + \ldots\ldots \frac{1}{x^3}} = \frac{1}{3 \cdot 8} \times \overline{B - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{4n^4} + \frac{1}{12n^6} - \text{\&c.}}$, and ſo on for the other ſerieſes: hence, by ſubſtitution, hyp. log. $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. .... $\frac{2x}{2x-1} = \frac{1}{2}$ hyp. log. $x -$ $,12078223764 + \frac{1}{8n} + \frac{1}{8n^2} + \frac{23}{192n^3} + \frac{7}{64n^4} + \frac{61}{640n^5}$ + &c.; conſequently $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. .... $\frac{2x}{2x-1}$ = the natural number correſponding to the right hand ſide of the equation; hence $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ .... $\frac{2x}{2x-1}$ = twice that natural number.

Ex. Let $x = 10000$; then

$$
\begin{array}{rr}
\tfrac{1}{2}\text{ hyp. log. } x = & 4,605170185988 \\
\text{conſt. quant.} = & ,120782237640 \\
\hline
 & 4,484387948348 \\
\frac{1}{8n} = & ,000012498750 \\
\frac{1}{8n^2} = & ,000000001249 \\
\hline
 & \underline{4,484400448347}
\end{array}
$$

the natural number correſponding to which hyp. log. is 88,6238, &c., conſequently $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ ...... $\frac{20000}{19999} = 177,2476$.

If x be a very large number, it may be ſufficiently exact in moſt caſes to take twice the natural number correſponding to the hyp. log. of $\frac{1}{2}$ hyp. log. $x - ,120782237640$.

* * *